CONTACT PROBLEM FOR A LAYERED INHOMOGENEOUS AGING CYLINDER REINFORCED BY A RIGID RING

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UDC 539.376

1. Problem Statement. We examine the contact interaction of rigid supporting ring with a two-layer viscoelastic inhomogeneous cylinder acted upon by an external pressure. We assume that the layers of the cylinder are made of different materials at different times. The external layer, of inner radius a and outer radius b, is made at the time τ_1. The internal layer, of thickness h, is made at the time τ_2. The value of h is much smaller than the characteristic size of the contact region ℓ and of the inner radius of the two-layer cylinder (a-h). The layers are in smooth contact. At the time τ_0 a rigid ring is mounted in the cylinder with tension δ_0. The surface profile of the ring is g(z). External pressure P(t) is then applied. The ring is at a sufficient distance from the cylinder ends for their effect on the stressed state under the ring itself to be neglected. The endfaces are stopped with rigid plugs that prevent axial movement (Fig. 1).

In cylindrical coordinates we have the following characteristic equations [1-3]:

$$\sigma_r^{(i)} = \frac{E_i(t-\tau_i)}{(1-2\nu_i)(1+\nu_i)}(\mathbf{I}+\mathbf{N}_i)\left[(1-\nu_i)\varepsilon_r^{(i)} + \nu_i\left(\varepsilon_\theta^{(i)} + \varepsilon_z^{(i)}\right)\right], \tag{1.1}$$

$$\sigma_\theta^{(i)} = \frac{E_i(t-\tau_i)}{(1-2\nu_i)(1+\nu_i)}(\mathbf{I}+\mathbf{N}_i)\left[(1-\nu_i)\varepsilon_\theta^{(i)} + \nu_i\left(\varepsilon_z^{(i)} + \varepsilon_r^{(i)}\right)\right],$$

$$\sigma_z^{(i)} = \frac{E_i(t-\tau_i)}{(1-2\nu_i)(1+\nu_i)}(\mathbf{I}+\mathbf{N}_i)\left[(1-\nu_i)\varepsilon_z^{(i)} + \nu_i\left(\varepsilon_r^{(i)} + \varepsilon_\theta^{(i)}\right)\right],$$

$$\tau_{rz}^{(i)} = \frac{E_i(t-\tau_i)}{1+\nu_i}(\mathbf{I}+\mathbf{N}_i)\varepsilon_{rz}^{(i)}, \quad (\mathbf{I}+\mathbf{N}_i)^{-1} = (\mathbf{I}-\mathbf{L}_i),$$

$$\mathbf{N}_i\omega(t) = \int_{\tau_0}^{t} \omega(\tau) R_i(t-\tau_i, \tau-\tau_i)\, d\tau,$$

$$\mathbf{L}_i\omega(t) = \int_{\tau_0}^{t} \omega(\tau) K_i(t-\tau_i, \tau-\tau_i)\, d\tau,$$

$$K_i(t,\tau) = E_i(\tau)\frac{\partial}{\partial\tau}\left[\frac{1}{E_i(\tau)} + C_i(t,\tau)\right],$$

where the subscript i = 1, 2 identifies the characteristics of the internal and external layers of the cylinder, respectively; $\sigma_r^{(i)} = \sigma_r^{(i)}(t, r, z)$, etc., are the components of the stress tensor; $\varepsilon_r^{(i)} = \varepsilon_r^{(i)}(r, z, t)$ are the components of the strain tensor; $E_i(t - \tau_i)$ are the instantaneous elastic strain moduli; ν_i are constant Poisson's ratios; $C_i(t, \tau)$ are the measures of creep of the cylinder layers; $K_i(t - \tau_i, \tau - \tau_i)$ are creep nuclei; $R_i(t - \tau_i, \tau - \tau_i)$ are their resolvents; r and z are the radial and axial coordinates; t is the current time; $\mathbf{I}$ is the identity operator.

Equilibrium equations and the relations of deformations and displacements also take place [$u_i = u_i(r, z, t)$, $w = w_i(r, z, t)$ are the radial and axial displacements, respectively]:

$$\partial\sigma_r^{(i)}/\partial r + \partial\tau_{rz}^{(i)}/\partial z + (\sigma_r^{(i)} - \sigma_\theta^{(i)})/r = 0, \qquad \partial\tau_{rz}^{(i)}/\partial r + \partial\sigma_z^{(i)}/\partial z + \tau_{rz}^{(i)}/r = 0; \tag{1.2}$$

$$\varepsilon_r^{(i)} = \partial u_i/\partial r,\ \varepsilon_\theta^{(i)} = u_i/r,\ \varepsilon_z^{(i)} = \partial w_i/\partial z, \qquad \varepsilon_{rz}^{(i)} = (1/2)(\partial u_i/\partial z + \partial w_i/\partial r). \tag{1.3}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 101-109, November-December, 1990. Original article submitted December 14, 1987; revision submitted June 14, 1989.

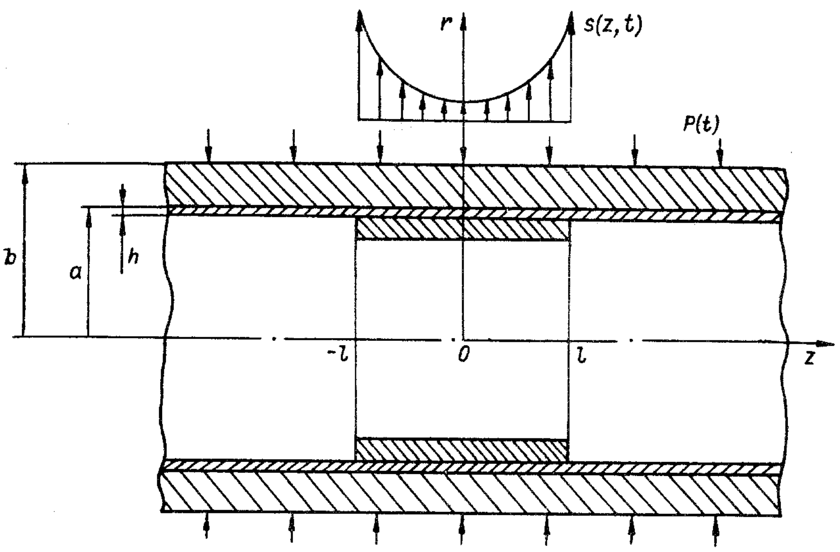

Fig. 1

2. Derivation of the Integral Equation for the Contact Problem. We first solve an auxiliary problem, substituting for the rigid ring a certain distributed load $p(z, t)$, which are nonzero, on the segment $|z| \leq \ell$. We will find the radial displacement of the inner surface of the two-layer cylinder, presenting this solution of the auxiliary problem as the result of superposition of solutions of two independent problems: A) describing the effect of the load upon an infinite two-layer cylinder in absence of an external pressure; and B) describing the plane strain of a two-layer cylinder acted upon by a uniform external pressure.

We first consider problem A. We add to relations (1.1)-(1.3) the boundary conditions

$$r = a - h\colon\ \sigma_r^{(1)} = p(z, t),\ \tau_{rz}^{(1)} = 0;$$
$$r = a\colon\ \sigma_r^{(1)} = \sigma_r^{(2)},\quad u_1 = u_2,\quad \tau_{rz}^{(1)} = \tau_{rz}^{(2)} = 0; \tag{2.1}$$
$$r = b\colon\ \sigma_r^{(2)} = 0,\ \tau_{rz}^{(2)} = 0\colon\ |z| \to \infty\colon\ \sigma_r^{(i)},\ \sigma_\theta^{(i)},\ \sigma_z^{(i)},\ \tau_{rz}^{(i)} \to 0.$$

In view of the relative smallness of the thickness of the inner layer ($h < \ell$) and assuming that either the compliances of the elements of the different layers are of the same order of magnitude or the compliance of the elements of the internal layer is greater, we expand $\tau_{rz}^{(1)}$ as a series in the powers of $(a - r)\ell^{-1}$ and limit the analysis to linear terms [3, 4]. Now, taking into account boundary conditions (2.1) for $\tau_{rz}^{(1)}$ at $r = a-h$ for $r = a$, we obtain $\tau_{rz}^{(1)} \equiv 0$. From the second equilibrium equation in (1.2) we have $\sigma_z^{(1)} = f(r, t)$. However, $\sigma_z^{(1)} \to 0$ as $|z| \to \infty$ [see (2.1)]. Therefore, $\sigma_z^{(1)} \equiv 0$. Hence, on the basis of (1.1) for $\sigma_z^{(1)}$ we find

$$\varepsilon_z^{(1)} = -\nu_1(1-\nu_1)^{-1}\left(\varepsilon_r^{(1)} + \varepsilon_\theta^{(1)}\right). \tag{2.2}$$

The first equilibrium equation in (1.2), taking into account (1.1), (2.2), and $\tau_{rz}^{(1)} \equiv 0$, appears as [the prime denotes the derivative with respect to r]

$$\left(\varepsilon_r^{(1)\prime} + \nu_1\varepsilon_\theta^{(1)\prime}\right)(1-\nu_1)^{-1} + \left(\varepsilon_r^{(1)} - \varepsilon_\theta^{(1)}\right)r^{-1} = 0. \tag{2.3}$$

Integrating (2.3), we establish with the aid of the identity $(\varepsilon_r^{(1)} - \varepsilon_\theta^{(1)})r^{-1} = \varepsilon_\theta^{(1)\prime}$ that $\varepsilon_r^{(1)} + \varepsilon_\theta^{(1)} = 2\Psi(z, t)$. By virtue of (1.3),

$$\partial u_1/\partial r + u_1/r = 2\Psi\ (z,\ t). \tag{2.4}$$

The solution of (2.4) can be written in this form:

$$u_1 = r\Psi\ (z,\ t) + \Phi\ (z,\ t)r^{-1}. \tag{2.5}$$

Splicing the radial displacements on the conjugation surfaces of layers [see (2.1)], we now obtain the expression

$$\Phi\ (z,\ t) = u_2\,(a,\ z,\ t)a - \Psi(z,\ t)\ a^2. \tag{2.6}$$

Substituting (2.6) into (2.5), we have

$$u_1 = \Psi(z, t)(r - a^2 r^{-1}) + u_2(a, z, t) a r^{-1}. \tag{2.7}$$

On the basis of (2.7), (1.3), and (2.2), we determine $\varepsilon_r^{(1)}$, $\varepsilon_\theta^{(1)}$, and $\varepsilon_z^{(1)}$. With the aid of (1.1), we obtain

$$\sigma_r^{(1)} = E_1 (1 - \nu_1^2)^{-1} (\mathbf{I} + \mathbf{N}_1) \{\Psi(z, t)[1 + \nu_1 + (1 - \nu_1) a^2 r^{-2}] - (1 - \nu_1) a r^{-2} u_2(a, z, t)\}. \tag{2.8}$$

Satisfying boundary condition (2.1) at $\sigma_r^{(1)}$ for $r = a - h$, taking into account (2.8) and ignoring values on the order of $h a^{-1}$ relative to unity, we find

$$\Psi(z, t) = (1/2)\left[(1 - \nu_1^2)(\mathbf{I} - \mathbf{L}_1) p(z, t) E_1^{-1} + (1 - \nu_1) u_2(a, z, t) a^{-1}\right]. \tag{2.9}$$

The radial displacement of the inner surface of the cylinder and the stress $\sigma_r^{(1)}$ at $r = a$, by virtue of (2.7)-(2.9), can be written as

$$\begin{gathered} u_1(a - h, z, t) = -(1 - \nu_1)^2 (\mathbf{I} - \mathbf{L}_1) p(z, t) E_1^{-1} h + u_2(a, z, t), \\ \sigma_r^{(1)}(a, z, t) = p(z, t), \end{gathered} \tag{2.10}$$

Here $u_2(a, z, t)$ is found from the solution of the problem for the external layer under the following boundary conditions [see (2.1) and (2.10) for $\sigma_r^{(1)}$]:

$$r = a\colon \sigma_r^{(2)} = p(z, t),\ \tau_{rz}^{(2)} = 0;\ r = b\colon \sigma_r^{(2)} = 0,\ \tau_{rz}^{(2)} = 0. \tag{2.11}$$

The stressed-strained state of the external layer is determined by the compatibility principle [5] and the instantaneous elastic solution of the problem constructed on the basis of Galerkin's representation [6]:

$$\varphi(r, z) = \int_{-\infty}^{\infty} [A(\alpha) I_0(\alpha r) + B(\alpha) \alpha r I_1(\alpha r) + C(\alpha) K_0(\alpha r) + D(\alpha) \alpha r K_1(\alpha r)] e^{i\alpha z} d\alpha,$$

$$\Delta\Delta\varphi = 0,\ \Delta = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2,$$

$$\sigma_r^{(2)} = \frac{\partial}{\partial z}\left[\nu_2 \Delta\varphi - \frac{\partial^2\varphi}{\partial r^2}\right],\ \sigma_z^{(2)} = \frac{\partial}{\partial z}\left[(2 - \nu_2)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right],$$

$$\sigma_\theta^{(2)} = \frac{\partial}{\partial z}\left[\nu_2 \Delta\varphi - \frac{1}{r}\frac{\partial\varphi}{\partial r}\right],\ \tau_{rz}^{(2)} = \frac{\partial}{\partial r}\left[(1 - \nu_2)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right],$$

$$u_2 = -\frac{1 + \nu_2}{E_2}\frac{\partial^2\varphi}{\partial r\,\partial z},\ w_2 = \frac{1 + \nu_2}{E_2}\left[2(1 - \nu_2)\Delta\varphi - \frac{\partial^2\varphi}{\partial z^2}\right],$$

where $I_0(\alpha r)$, $I_1(\alpha r)$, $K_0(\alpha r)$, $K_1(\alpha r)$ are the Bessel functions of imaginary arguments. The expression for the radial displacement of the inner surface appears as

$$u_2(a, z, t) = -\frac{2(1 - \nu_2^2)}{\pi}(\mathbf{I} - \mathbf{L}_2)\int_{-l}^{l} \frac{p(\xi, t)}{E_2} k(z, \xi)\, d\xi, \tag{2.12}$$

$$k(z, \xi) = \int_0^\infty \frac{L(\alpha)}{\alpha}\cos\alpha(z - \xi)\, d\alpha,\ L(\alpha) = \alpha\left[b^{-1} + b^0 D_1^2(\alpha) - \alpha^2 b C_1^2(\alpha)\right] S^{-1}(\alpha),$$

$$S(\alpha) = a^0 b^{-1} + b^0 a^{-1} + a^0 b^0 D_1^2(\alpha) - b^0 \alpha^2 a B_1^2(\alpha) + \alpha^4 a b A_1^2(\alpha) - a^0 b \alpha^2 C_1^2(\alpha),$$

$$A_1(\alpha) = I_0(\alpha a) K_0(\alpha b) - I_0(\alpha b) K_0(\alpha a),$$

$$B_1(\alpha) = I_0(\alpha a) K_1(\alpha b) + I_1(\alpha b) K_0(\alpha a),$$

$$C_1(\alpha) = I_0(\alpha b) K_1(\alpha a) + I_1(\alpha a) K_0(\alpha b),$$

$$D_1(\alpha) = I_1(\alpha a) K_1(\alpha b) - I_1(\alpha b) K_1(\alpha a),$$

$$a^0 = 2(1 - \nu_2) a^{-1} + a\alpha^2,\ b^0 = 2(1 - \nu_2) b^{-1} + b\alpha^2,$$

Here the kernel $k(z, \xi)$ retains all the basic characteristics of the kernels of plane contact problems [4].

Thus, the radial displacement of the inner surface of a two-layer cylinder in problem A is expressed by the following formula [see (2.10)-(2.12)]:

$$u_1(a - h, z, t) = -(1 - \nu_1^2) h (\mathbf{I} - \mathbf{L}_1) p(z, t) E_1^{-1}(t - \tau_1) - \frac{2(1 - \nu_2^2)}{\pi}(\mathbf{I} - \mathbf{L}_2)\int_{-l}^{l} \frac{p(\xi, t)}{E_2(t - \tau_2)} k(z, \xi)\, d\xi. \tag{2.13}$$

The plane strain of the cylinder in problem B is investigated under the boundary conditions

$$r=a-h:\ \sigma_r^{(1)}=0;\ r=a:\ \sigma_r^{(1)}=\sigma_r^{(2)},\ u_1=u_2;\ r=b:\ \sigma_r^{(2)}=-P(t).$$

The closed solution of this problem allows one to write the displacement u_1 at $r = a - h$:

$$\begin{gathered}
u_1(a-h,z,t)=-(\mathbf{I}-\mathbf{L}_1)(\mathbf{I}+\mathbf{N}_3)\theta_1(t)(\mathbf{I}-\mathbf{L}_2)P(t)E_2^{-1}(t-\tau_2),\\
\theta_1(t)=b_1\theta(t),\ \theta(t)=\left[b_2E_1(t-\tau_1)E_2^{-1}(t-\tau_2)+b_3\right]^{-1},\\
b_1=2(1-\nu_1)(a-h)^{-1}(1+\nu_2)\{(1-2\nu_2)a+\\
+a^2b^2(b^2-a^2)^{-1}[(1-2\nu_2)ab^{-2}+a^{-1}]\},\\
b_2=(1+\nu_1)^{-1}[a^{-2}-(a-h)^{-2}](1+\nu_2)a^2b^2\times\\
\times(b^2-a^2)^{-1}[(1-2\nu_2)ab^{-2}+a^{-1}],\\
b_3=(1-2\nu_1)(a-h)^{-2}a+a^{-1},\ \mathbf{N}_3\omega(t)=\int\limits_{\tau_0}^{t}\omega(\tau)R_3(t,\tau)\,d\tau,
\end{gathered}\tag{2.14}$$

where $R_3(t, \tau)$ is the resolvent of the kernel $K_3(t, \tau) = [b_2K_2(t - \tau_2, \tau - \tau_2)E_1(\tau - \tau_1) \times E_2^{-1}(\tau - \tau_2) + b_3K_1(t - \tau_1, \tau - \tau_1)]\theta(t)$.

The radial displacement of the inner surface of a two-layer cylinder in an auxiliary problem of the effect of a normal distributed load $p(z, t)$ and the external pressure $P(t)$ applied to the cylinder is obtained by taking the sum of (2.13) and (2.14).

Setting $p(z, t) = -s(z, t)$ and equating the resulting displacement to the tension of the ring, with adjustment for the profile of the ring surface $\delta_0 - g(z)$, we write the integral equation for the contact problem:

$$\begin{gathered}
\left(1-\nu_1^2\right)h(\mathbf{I}-\mathbf{L}_1)\frac{s(z,t)}{E_1(t-\tau_1)}+\frac{2\left(1-\nu_2^2\right)}{\pi}(\mathbf{I}-\mathbf{L}_2)\int\limits_{-l}^{l}\frac{p(\xi,t)}{E_2(t-\tau_2)}\times\\
\times k(z,\xi)\,d\xi=(\mathbf{I}-\mathbf{L}_1)(\mathbf{I}+\mathbf{N}_3)\theta_1(t)(\mathbf{I}-\mathbf{L}_2)P(t)E_2^{-1}(t-\tau_2)+\delta_0-g(z)\quad(|z|\leqslant l).
\end{gathered}\tag{2.15}$$

3. Solution of the Integral Equation of the Contact Problem. In (2.1) we change the variables according to the following formulas

$$\begin{gathered}
z^*=zl^{-1},\ \xi^*=\xi l^{-1},\ t^*=t\tau_0^{-1},\ \tau^*=\tau\tau_0^{-1},\\
\tau_1^*=\tau_1\tau_0^{-1},\ \tau_2^*=\tau_2\tau_0^{-1},\ \delta_0^*=\delta_0l^{-1},\ k^*(z^*,\xi^*)=\pi^{-1}k(z,\xi),\\
q^*(z^*,t^*)=2\left(1-\nu_2^2\right)q(z,t)E_2^{-1}(t-\tau_2),\ g^*(z^*)=g(z)l^{-1},\\
P^*(t^*)=2\left(1-\nu_2^2\right)P(t)E_2^{-1}(t-\tau_2),\ \theta_1^*(t^*)=(1/2)\theta_1(t)l^{-1}\left(1-\nu_2^2\right)^{-1},\\
c(t^*)=\frac{\left(1-\nu_1^2\right)E_2(t-\tau_2)h}{2\left(1-\nu_2^2\right)E_1(t-\tau_1)l},\\
K^{(1)}(t^*,\tau^*)=\frac{E_1(t-\tau_1)E_2(\tau-\tau_2)}{E_1(\tau-\tau_1)E_2(t-\tau_2)}K_1(t-\tau_1,\tau-\tau_1)\tau_0,\\
K^{(2)}(t^*,\tau^*)=K_2(t-\tau_2,\tau-\tau_2)\tau_0,\\
K^{(0)}(t^*,\tau^*)=K_1(t-\tau_1,\tau-\tau_1)\tau_0,\\
f^*(z^*,t^*)=\delta^*(t^*)-g^*(z^*),\ R_3^*(t^*,\tau^*)=R_3(t,\tau)\tau_0,\\
\delta^*(t^*)=\delta_0^*+\left(\mathbf{I}-\mathbf{L}_0^*\right)(\mathbf{I}+\mathbf{N}^*)\theta_1^*(t^*)\left(\mathbf{I}-\mathbf{L}_2^*\right)P^*(t^*),\\
\mathbf{L}_i^*\omega(t)=\int\limits_1^t\omega(\tau)K^{(i)}(t,\tau)\,d\tau\quad(i=0,1,2),\ \mathbf{N}^*\omega(t)=\int\limits_1^t\omega(\tau)R_3^*(t,\tau)\,d\tau,\\
\mathbf{A}^*p(z^*,t^*)=\int\limits_{-1}^{1}p(\xi^*,t^*)k^*(z^*,\xi^*)\,d\xi^*.
\end{gathered}$$

Omitting the asterisks in the notations for all quantities except for operators, we write the resolving equation of the problem as

$$c(t)\left(\mathbf{I}-\mathbf{L}_1^*\right)s(z,t)+\left(\mathbf{I}-\mathbf{L}_2^*\right)\mathbf{A}^*s(z,t)=f(z,t),\tag{3.1}$$

where the function of contact pressures $s(z, t)$ and the right-hand side of the equation $f(z, t)$ are continuous with respect to time in $L_2[-1, 1]$; $c(t) > 0$ is a continuous function

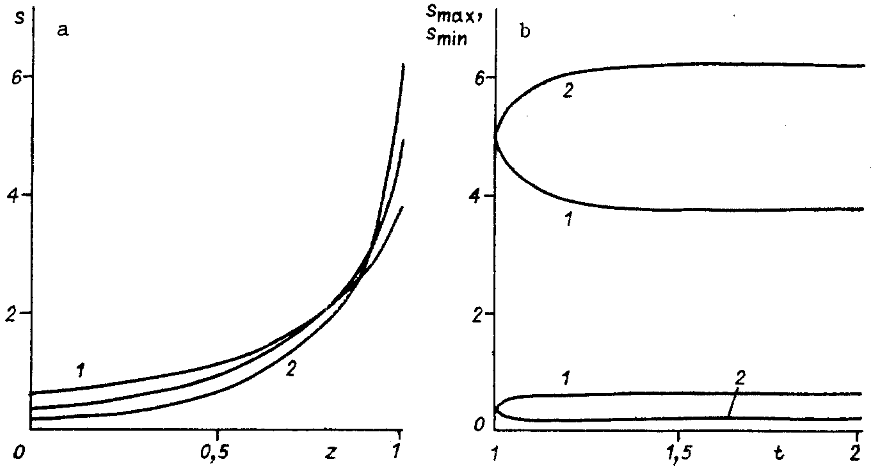

Fig. 2

of t; the kernels of Volterra operators L_i^* (i = 1, 2) are either continuous or weakly singular. The operator A* is fully continuous, self-conjugate, and positively defined from $L_2[-1, 1]$ in $L_2[-1, 1]$, where

$$\int_{-1}^{1}\int_{-1}^{1} k^2(z, \xi)\, d\xi\, dz < \infty.$$

We represent the solution of (3.1) as a series in eigenfunctions (z) of the operator $\mathbf{A}^*$, which correspond to its eigenvalues α_i (i = 0, 1, 2, ...) [7, 8]:

$$s(z, t) = \sum_{i=0}^{\infty} \omega_i(t)\varphi_i(z), \quad f(z, t) = \sum_{i=0}^{\infty} f_i(t)\varphi_i(z); \tag{3.2}$$

$$\mathbf{A}^*\varphi_i(z) = \alpha_i\varphi_i(z). \tag{3.3}$$

Substituting (3.2) into (3.1), we obtain

$$\omega_i(t) = \frac{f_i(t)}{c(t)+\alpha_i} + \int_1^t \frac{f_i(\tau)}{c(\tau)+\alpha_i} R^i(t, \tau)\, d\tau$$

{$R^i(t, \tau)$ is the resolvent of the kernel $K^i(t, \tau) = [c(t)K^{(1)}(t, \tau) + \alpha_i K^{(2)}(t, \tau)]\cdot[c(t) + \alpha_i]^{-1}$}.

We have thus constructed the solution. It is worthwhile to briefly describe a procedure for finding eigenfunctions and eigenvalues of the operator $\mathbf{A}^*$. We take eigenfunctions and the kernel of the operator $\mathbf{A}^*$ in the form {$P_k^*(z)$ (k = 0, 1, ...) is the basis of $L_2 \times [-1, 1]$}:

$$\varphi_i(z) = \sum_{\kappa=0}^{\infty} a_\kappa^i P_\kappa^*(z), \quad k(z, \xi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r_{mn} P_m^*(z) P_n^*(\xi). \tag{3.4}$$

Now, on the basis of (3.3) and taking into account (3.4), we write a system of algebraic equations for determining the eigenvalues α_i and the coefficients of expansion of the eigenfunctions a_k^i:

$$\sum_{n=0}^{\infty} r_{mn} a_n^i = \alpha_i a_m^i \quad (m = 0, 1, 2, \ldots).$$

Limiting the analysis to N terms of the basis $P_k^*(z)$, we obtain the N-th approximation of the Bubnov—Galerkin method [9].

When the layers of the cylinder are made of the same aging viscoelastic material at the same point in time, the pretightening of the ring $\delta_0 = 0$ and the base profile is described by the function g(z) = 0. The creep has no effect on the stressed state of the cylinder. The problem solution coincides with the elastic one.

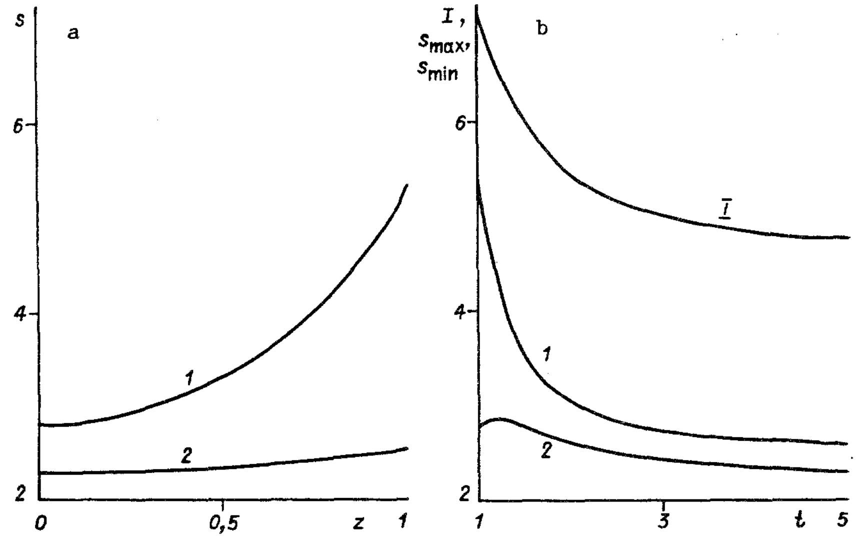

Fig. 3

In conclusion, we will discuss some specifics of the numeric determination of the coefficients of expansion r_{mn} of the contact problem kernel $k(z, \xi)$. Taking for the basis an orthonormal Legendre polynomial, we obtain, according to [4], expressions of this form:

$$r_{mn}(\lambda) = (-1)^{m+n}[(4m+1)(4n+1)]^{1/2}\lambda \int_0^\infty \frac{L(u)}{u^2} J_{1/2+2m}\left(\frac{u}{\lambda}\right) J_{1/2+2n}\left(\frac{u}{\lambda}\right) du \tag{3.5}$$

$$(m,\ n = 0,\ 1,\ 2,\ \ldots).$$

Calculation of integrals (3.5) at large m and n requires special analysis, because the integrands oscillate rapidly. An effective numeric algorithm can be constructed on the basis of [10]. We substitute the integrand as the product of the slow-varying and fast-varying functions. Suppose that for the fast-varying function K(x) the integral is taken in an explicit form. Dividing the integration limits into n parts and approximating the slow-varying function f(x) on each interval by its value at the center, we obtain

$$\int_a^b f(x)\,K(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,K(x)\,dx \approx \sum_{i=1}^{n} f_i\,(K_i - K_{i-1}),$$

where $x_0 = a$, $x_n = b$, K_i is the primitive function K(x) at the point x_i, $f_i = f[(x_i + x_{i-1})/2]$. All the necessary expressions for the primitive fast-oscillating integrand functions from (3.5) can be found in [11]. Besides, with the above algorithm we can readily determine the finite value of the upper limit in (3.5), which satisfies the desired accuracy of calculation of the coefficients r_{mn} because the behavior of the function L(u), as $u \to \infty$, can easily be analyzed.

4. Example. We will consider a cylinder with inner and outer layers of aging viscoelastic material. The elastic characteristics E_i, ν_i ($i = 1, 2$) are constant. A measure of the creep of the material is expressed by [5]

$$C_1(t,\ \tau) = C_2(t,\ \tau) = C(t,\ \tau) = [C_0 + A_0\exp(-\beta\tau)]\ \{1 - \exp[-\gamma(t - \tau)]\}. \tag{4.1}$$

We proceed from the following values of the parameters of the two-layer cylinder [5, 12]: $E_1 = E_2 = 5\cdot 10^3$ MPa, $\nu_1 = \nu_2 = 0.1$, $C_0E_1 = 0.5522$, $A_0E_1 = 4$, $ab^{-1} = 0.81$, $\beta = 0.031$ day^{-1}, $\gamma = 0.06$ day^{-1}, $h\ell^{-1} = 0.31$, $b\ell^{-1} = 10$, $c(t) = 0.155$, $P(t) = 1$, $g(z) = 0$, $\delta_0 = 0$, $\theta_1(t) = 20.72$.

Suppose that the outer layer is made at time zero and the inner layer 50 days later. The outer pressure is applied 65 days later (for dimensionless values $\tau_2 = 0$, $\tau_1 = 0.77$). The curves in the figures for this case are identified by the index 1. We will also examine the variant where the inner layer is made at time zero and the outer layer 50 days later, with the same load application time $\tau_0 = 65$ days, i.e., $\tau_1 = 0$, $\tau_2 = 0.77$. The curves are identified by the index 2.

Figure 2a illustrates the distribution of contact pressures under the ring for the two cases at different points in time. The unmarked curves show the stress distribution at the time when the external pressure is applied. Curves 1 and 2 describe the distribution of contact pressures at t = 2. Figure 2b illustrates the variations of maximal and minimal contact pressures over time in these two cases.

The plots show that, if the outer layer is younger than the inner layer, the stressed state under the ring levels out over time. In the opposite case, the nonuniformity increases. The integral characteristic of contact pressures $I(t) = \int_{-1}^{1} s(z, t)dz$ decreases slightly over time; for engineering calculations one can assume with an acceptable accuracy that $I(t) = I(1)$. The effect of redistribution of contact pressures is the principle factor in this case.

Let us now consider a cylinder with an outer layer made of a contact material and an inner layer of an aging viscoelastic material: $E_1E_2^{-1} = 0.025$, $\nu_1 = 0.1$, $\nu_2 = 0.3$, $ab^{-1} = 0.81$, $C_0E_1 = 0.5522$, $A_0E_1 = 4$, $\tau_0 = 15$ days, $\beta = 0.031$ day^{-1}, $\gamma = 0.06$ day^{-1}, $h\ell^{-1} = 0.06$, $b\ell^{-1} = 10$, $c(t) = 1.36$, $\theta(t) = 24.06$, $\tau_1 = 0$, $P(t) = 1$, $g(z) = 0$, $\delta_0 = 0$.

Figure 3a shows the distribution of contact pressures under the ring at the time of application of the external pressure t = 1 (curve 1) and at t = 5 (2). Figure 3b describes the variations of maximum (1) and minimal (2) stresses under the ring and the integral characteristic I(t) of contact pressures over time. We see that over time the stress distribution is evened out substantially and the stresses are considerably relaxed. Calculations reveal two tendencies in the formation of the stressed state under the ring: a tendency for a change in the contact stresses due to the inhomogeneity of the cylinder, and a tendency for a decrease in stresses by relaxation. The former tendency can manifest itself in two ways depending on the type of inhomogeneity. It is clearly seen in the first example, where the effect of relaxations is small. The second example illustrates the interaction of the two tendencies. On one hand, the inhomogeneity of the material smoothes the distribution of contact stresses, with the maximal stresses diminishing and minimal stresses growing. On the other hand, contact stresses are reduced by relaxation. For maximal stresses both tendencies lead to a reduction in stresses. For minimal stresses, initially the former tendency is dominant, and stresses grow slightly. However, over time, the second tendency prevails and minimal stresses begin to decrease.

The authors thank N. Kh. Arutyunyan for his assistance and interest in this work.

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